

# The Dispersion Relation for the $1/\sinh^2$ Potential in the Classical Limit

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## **Abstract:**

The dispersion relation for the inverse hyperbolic potential is calculated in the classical limit. This is shown for both the low amplitude phonon branch and the high amplitude soliton branch. It is shown these results qualitatively follow that previously found for the inverse squared potential where explicit analytic solutions are known.

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## Introduction

In one spatial dimension, a class of integrable many-body systems is the Calogero-Sutherland-Moser systems. They consist of many identical non-relativistic particles interacting through two-body potentials of the inverse square type and various extensions such as the inverse sine squared, the inverse hyperbolic squared, and the inverse squared Jacobian elliptic function<sup>(1,2,3)</sup>. The significance of these models is they represent a class of integrable many-body systems. One test for integrability is to use the Lax method, which is the method first applied by Calogero, et. al to the inverse squared potential<sup>(4)</sup>. Using this method one attempts to show that for certain potentials one can find two Hermitean  $N \times N$  matrices  $L$  and  $A$  that follow the Lax equation  $dL/dt=i(AL-LA)$ . With this  $L$  evolves as a transformation generated by  $A$  and  $\det[L-\lambda I]$  is a constant of motion.

In this paper we look at the inverse hyperbolic squared potential and derive the dispersion relation in the classical limit. This is useful because it is one method one can compare the quantum mechanical system with the purely classical many-body soliton system, which may also be derived using the classical equations of motion. This comparison was done in a previous paper for the inverse squared potential<sup>(5)</sup>. However, the dispersion relation for this system is a bit more problematic as no closed form analytic solution is known.

## Dispersion relation for the inverse hyperbolic potential

We begin with the many body Hamiltonian

$$H = -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \frac{g\alpha^2}{2} \sum_{j>i=1}^N \frac{1}{\sinh^2[\alpha(x_j - x_i)]}. \quad (1)$$

We make a change of variable,  $x \rightarrow \alpha x$ , so that  $1/\alpha$  is the unit of length, and so the Hamiltonian becomes

$$H = -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \lambda(\lambda-1) \sum_{j>i=1}^N \frac{1}{\sinh^2[x_j - x_i]}, \quad (2)$$

where energies are measured in units of  $\hbar^2 \alpha^2 / m$ , and  $\lambda(\lambda-1) = gm / 2\hbar^2$ . Thus, we see that the classical limit  $\hbar \rightarrow 0$  corresponds to  $\lambda \rightarrow \infty$ ; this is the limit we investigate in this paper. This quantum system was originally solved by Sutherland<sup>(1,2,3)</sup>, using the asymptotic Bethe ansatz<sup>(1,6,7)</sup>, and a proof of integrability due to Calogero, et. al.<sup>(4)</sup>

As is well known<sup>(1,6,7)</sup>, the solution by the asymptotic Bethe ansatz for the low energy properties makes use of two quantities  $\rho(k)$  and  $\varepsilon(k)$ , which are solutions of the following integral equations:

$$\rho(k) + \frac{1}{2\pi} \int_{-B}^B \theta'(k - k') \rho(k') dk' = \frac{1}{2\pi}, \quad (3)$$

and

$$\varepsilon(k) + \frac{1}{2\pi} \int_{-B}^B \theta'(k - k') \varepsilon(k') dk' = \frac{k^2}{2} - \mu. \quad (4)$$

The limit to the integral  $B$  is related to the density  $d = N/L$  through

$$\int_{-B}^B \rho(k) dk = d, \quad (5)$$

while the chemical potential  $\mu$  is determined by the requirement that  $\varepsilon(\pm B) = 0$ . The kernel  $\theta'(k)$  of the integral equation is the derivative of the two-body phase shift  $\theta(k)$ , which for our (reduced) Hamiltonian is

$$\theta(k) = i \log \left[ \frac{\Gamma(1 + ik/2) \Gamma(\lambda - ik/2)}{\Gamma(1 - ik/2) \Gamma(\lambda + ik/2)} \right]. \quad (6)$$

The low-lying or zero temperature physical properties are determined from  $\rho(k)$  and  $\varepsilon(k)$  as follows. First, the ground state energy is calculated as

$$E/L = \frac{1}{2} \int_{-B}^B \rho(k) k^2 dk. \quad (7)$$

From this, the zero temperature equation of state may be determined. Second, the dispersion relation for low-lying excitations may be determined parametrically, with the energy given by  $|\varepsilon(k)|$ , and the group velocity  $v(k)$  by  $v(k) = \varepsilon'(k)/2\pi\rho(k)$ . (We choose the group velocity rather than the momentum to make the classical limit easier.) The dispersion relation has two branches:  $|k| \leq B$ , called the hole or phonon branch; and  $|k| \geq B$ , called the particle or soliton branch. This is discussed by Sutherland<sup>(1)</sup>, where the quantum Toda lattice is first solved, and the classical limit taken to recover the original results of Toda<sup>(7)</sup>.

In the classical limit  $\lambda \rightarrow \infty$ , we find  $B \rightarrow \infty$ , and so we rescale  $k$  by  $k = 2\lambda x$ . Then, the kernel of the integral equation becomes

$$\theta'(k) \rightarrow \frac{1}{2} \log \left[ 1 + 1/x^2 \right]. \quad (8)$$

Letting  $B = 2\lambda b$ , and keeping terms of leading order in  $\lambda$ , we find that for  $|x| \leq b$ ,  $\rho$  and  $\varepsilon$  obey the integral equations

$$\int_{-b}^b dy \log \left[ 1 + 1/(x-y)^2 \right] \lambda \rho(2\lambda y) = 1, \quad (9)$$

and

$$\int_{-b}^b dy \log \left[ 1 + 1/(x-y)^2 \right] \lambda^{-1} \varepsilon(2\lambda y) = 4\pi x^2 - 2\pi\mu/\lambda^2. \quad (10)$$

We see that the quantities that approach finite limits in the classical limit are  $\lambda\rho(2\lambda x)=\rho(x)$  and  $\lambda^{-1}\varepsilon(2\lambda x)=\varepsilon(x)$ , for  $|x|\leq b$ . Outside the integration region, when  $|x|\geq b$ , we use the full integral equations, so that

$$\rho(2\lambda x) = \frac{1}{2\pi} - \frac{1}{2\pi} \int_{-b}^b dy \log\left[1 + 1/(x-y)^2\right] \rho(y) \quad (11)$$

and

$$\lambda^{-2}\varepsilon(2\lambda x) = 2k^2 - \frac{\mu}{\lambda^2} - \frac{1}{2\pi} \int_{-b}^b dy \log\left[1 + 1/(x-y)^2\right] \varepsilon(y). \quad (12)$$

Thus, on the soliton branch, we define the quantities which approach finite classical limits as  $\rho(2\lambda x)=\rho(x)$  and  $\lambda^{-2}\varepsilon(2\lambda x)=\varepsilon(x)$ , for  $|x|\geq b$ .

Looking at the group velocity,  $v(k)=\varepsilon'(k)/2\pi\rho(k)$ , and using  $\varepsilon'(2\lambda x)=\varepsilon'(x)/2$ ,  $|x|\leq b$ , or  $\lambda^{-1}\varepsilon'(2\lambda x)=\varepsilon'(x)/2$ ,  $|x|\geq b$ , we see that  $\lambda^{-1}v(2\lambda x)=\varepsilon'(x)/4\pi\rho(x)=v(x)$ , for all  $x$ . Thus, the dispersion relation is given in the classical limit by velocity  $\lambda v(x)$  and energy  $\lambda|\varepsilon(x)|$ ,  $|x|\leq b$ , or  $\lambda^2|\varepsilon(x)|$ ,  $|x|\geq b$ . This gives an energy proportional to  $\hbar$  for the phonon branch, as is to be expected.

We thus see that the dispersion relation in the classical limit depends on the solution of the two integral equations

$$\int_{-b}^b dy \log\left[1 + 1/(x-y)^2\right] \rho(y) = 1 \quad (13)$$

and

$$\int_{-b}^b dy \log\left[1 + 1/(x-y)^2\right] \varepsilon(y) = 4\pi x^2 - 2\pi\mu. \quad (14)$$

In the second equation,  $\mu$  represents the old  $\mu/\lambda^2$ . The phonon branch of the dispersion curve can be found by a straightforward harmonic approximation

to the equations of motion. By familiar methods, this gives for the frequency  $\omega(k)$  of an oscillation of wavenumber  $k$ ,

$$\omega^2(k) = 4\lambda^2 \sum_{j=1}^{\infty} [1 - \cos(kj/d)] \frac{3 + 2 \sinh^2(j/d)}{\sinh^4(j/d)}. \quad (15)$$

Here,  $d$  is again the density  $N/L$ . The group velocity is given by the usual expression  $v(k) = \partial\omega(k)/\partial k$ , so the dispersion relation for the phonon branch  $\omega(v)$  at density  $d$  is thus determined parametrically. This must coincide with the previous expression when  $|v| \leq v_s = v(0)$ .

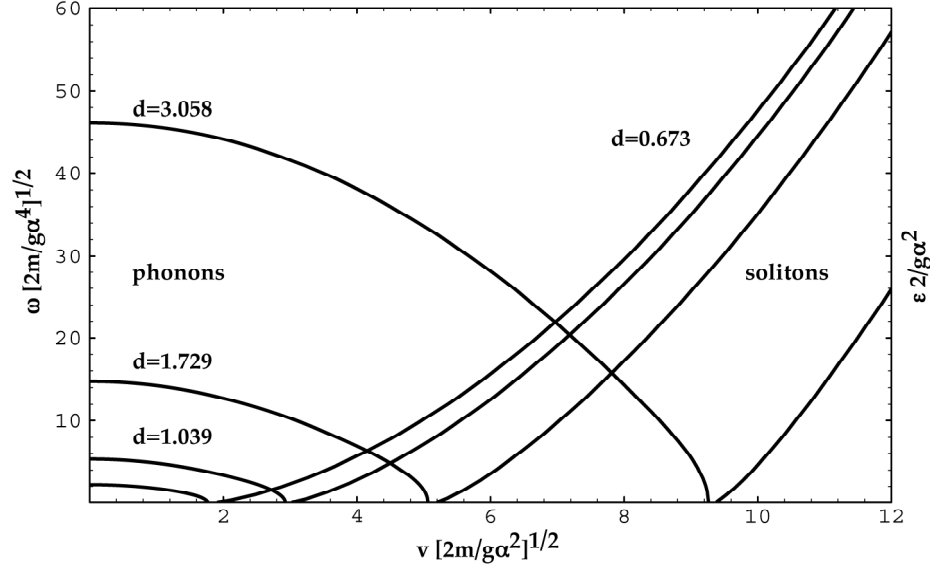
To check this correspondence, we make what are apparently convergent expansions for both  $\rho(x)$  and  $\varepsilon(x)$ ,

$$\rho(x) = \frac{d}{2\pi\sqrt{b^2 - x^2}} \left[ 1 + \sum_{j=1}^{\infty} \rho_j T_{2j}(x/b) \right] \quad (16)$$

and

$$\varepsilon(x) = \sqrt{b^2 - x^2} \sum_{j=0}^{\infty} \varepsilon_j U_{2j}(x/b). \quad (17)$$

Here  $T_j(x)$  and  $U_j(x)$  are Chebyshev polynomials of the first and second kind respectively. Using these expansions, which appear rapidly convergent except for the limit  $b \rightarrow \infty$ , corresponding to the inverse square potential, which can independently be evaluated<sup>(8,9)</sup>, one finds the dispersion relations shown in figure (1) for selected values of the density. This agrees numerically for the phonon branch with the previous harmonic approximation.



**Figure 1.** We show the dispersion relations for selected values of the density  $d = 3.058\dots, 1.729\dots, 1.039\dots$ , and  $0.673\dots$ . These values interpolate between the inverse square interaction and the Toda lattice. To the left we show the phonon dispersion relation by plotting the frequency as a function of the group velocity, and to the right we show the soliton dispersion relation by plotting the energy of the soliton as a function of the soliton velocity.

## Discussion

The dispersion relation qualitatively follows that found previously for the inverse squared potential in the classical limit<sup>(5)</sup>. In that case simple analytic results were found. For each density shown, the point where the curve for both the soliton branch and the phonon branch is zero represent where the velocity is the speed of sound for that density. The phonon branch represents simple low amplitude sound waves whereas the soliton branch represents high amplitude non-linear waves. We have found these results by taking the classical limit of a quantum mechanical system. It would be interesting to see how this compares to the direct solution of the classical equations of motion as we have done previously for the inverse squared potential<sup>(5)</sup>. We plan to investigate this in a future work.

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## References:

- 1) B.Sutherland, Rocky Mtn. J. Math. **8**, 413 (1978).
- 2) B.Sutherland and R.Römer, Phys. Rev. Lett. **71**, 2789 (1993).
- 3) B.Sutherland, R.Römer and B.S.Shastry, Phys. Rev. Lett. **73**, 2154 (1994).
- 4) F.Calogero, O.Ragnisco and C.Marchioro, Lett. Nuovo Cimento **13**, 383 (1975).
- 5) B.Sutherland and J.Campbell, Phys. Rev. B **50**, 888 (1994).
- 6) B.Sutherland, in *Exactly Solved Problems in Condensed Matter and Relativistic Field Theory*, ed. B.S.Shastry, S.S.Jha and V.Singh (Springer, Berlin, 1985).
- 7) B.Sutherland, in *Festschrift Volume for C.N. Yang's 70th Birthday*, ed. S.T.Yau (International, Singapore, 1995).
- 8) M.Toda, *Theory of Nonlinear Lattices*, 2nd ed. (Springer, Berlin, 1988).
- 9) B.Sutherland, J. Math. Phys. **12**, 251 (1971).